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AN AXIOMATIC APPROACH IN MINIMUM COST SPANNING TREE PROBLEMS WITH GROUPS

Gustavo Bergantiños[1] and María Gómez-Rúa[1]

[1]Research Group in Economic Analysis. Universidade de Vigo (Spain).

ABSTRACT

We study minimum cost spanning tree problems with groups, where agents are located in different villages, cities, etc. The groups are formed by agents living in the same village. In Bergantiños and Gómez-Rúa (2010, Economic Theory) we define the rule F as the Owen value of the irreducible game with groups and we prove that F generalizes the folk rule of minimum cost spanning tree problems. Bergantiños and Vidal-Puga (2007, Journal of Economic Theory) give two characterizations of the folk rule. In the first one they characterize it as the unique rule satisfying cost monotonicity, population monotonicity and equal share of extra costs. In the second characterization of the folk rule they replace cost monotonicity by independence of irrelevant trees and population monotonicity by separability. In this paper we extend such characterizations to our setting. Some of the properties are the same (cost monotonicity and independence of irrelevant trees) and the other need to be adapted. In general, we do it by claiming the property twice: once among the groups and the other among the agents inside the same group.

Keywords: minimum cost spanning tree problems, folk rule, cost sharing, axiomatization.

1. INTRODUCTION

In minimum cost spanning tree problems, *mcstp*, a group of agents want to be connected to a single supplier of some service. For instance, several villagers wish to construct pipes from their respective houses to a water supplier. Other examples are communication networks, such as Internet, telephone, or cable television. Usually, agents can reduce the total cost if some agents connect to the supplier through other agents. The cheapest graph connecting all agents to the supplier is called the minimum cost spanning tree. It is assumed that agents construct a minimum cost spanning tree. Now a cost allocation problem arises. The most relevant question is how to divide the cost of the minimum cost spanning tree among the agents.

The most popular way to answer this question is through the axiomatic characterization of a single rule (see for instance, Kar (2002), Dutta and Kar (2004), and Branzei *et al*. (2004)) or a family of rules (see for instance, Bergantiños *et al*. (2009), Bogomolnaia and Moulin (2010), and Chun and Lee (2009)).

Sometimes agents are located in different villages (see for instance, Dutta and Kar (2004) or Bergantiños and Lorenzo (2004, 2005, 2008)). This means, in terms of the cost matrix, that the connection cost between two agents of the same village is not larger than the connection cost between an agent of this village and an agent from other village. The classical model of *mcstp* includes these situations. Nevertheless, it ignores the fact that some group of agents are located in the same village. In Bergantiños and Gómez-Rúa (2010) we introduce a new class of problems which includes this fact in the model. We do it by considering an extra element in the model, namely a partition $G = \left\{G^1, ..., G^m\right\}$ of the set of agents N. For each $k = 1, ..., m$, G^k represents the group of agents located in the same village, city, ... We call these problems *mcstp* with groups.

The structure of groups or coalitions has been widely studied in the literature of cooperative games, through transferable utility (TU, for short) games in which bargaining occurs between groups and between players in the same group. The main idea is that the groups play among themselves as individual players in a game among groups, and then, the benefit obtained by each group is distributed among its members. Owen (1977) and Hart and Kurz (1983) studied the coalitional value that arises from applying the Shapley value (Shapley, 1953) twice: first in the game among groups, and then in a reduced game inside each group. There exist many other papers studying coalitional values, such as: Albizuri (2009), Pulido and Sánchez-Soriano (2009) and Gómez-Rúa and Vidal-Puga (2010) among others.

A very well-known rule in *mcstp* is the folk rule. It was introduced in Feltkamp *et al*. (1994) and studied later in further papers, for instance Branzei *et al*. (2004) and Bergantiños and Vidal-Puga (2007a, 2007b, 2009). The folk rule can be defined in different ways, for instance, through Kruskal´s algorithm (Kruskal, 1956) or as the Shapley value of the irreducible game or the optimistic game, which are TU games associated to the *mcstp* (see Bergantiños and Vidal-Puga (2007a, 2007b)). In Bergantiños and Gómez-Rúa (2010) we define the rule F in *mcstp* with groups as the Owen value of the irreducible game with groups. Since the Owen value generalizes the Shapley value, F coincides with the folk rule in classical *mcstp*.

Bergantiños and Vidal-Puga (2009) give an axiomatic characterization of the folk rule. In Bergantiños and Gómez-Rúa (2010) we extend it to *mcstp* with groups. Some of the properties are the same and other need to be adapted. We do it by claiming the property twice: once among the groups and the other among the agents inside the same group. Let us clarify this idea with the property of population monotonicity. In classical *mcstp*, population monotonicity says that if a new agent joins the society, no agent of the initial society can be worse off. In *mcstp* with groups population monotonicity is divided in two parts: population monotonicity over groups and population monotonicity over agents. Population monotonicity over groups says that if a new group joins the society, no agent of the initial society can be worse off. Population monotonicity over agents says that if agent i enters in group G^k, no agent of group G^k can be worse off. Moreover, if the connection costs between group G^k and the other groups do not change, agents of the other groups must pay the same.

Bergantiños and Vidal-Puga (2007a) give two characterizations of the folk rule. In the first one they prove that it is the unique rule satisfying cost monotonicity[1], population monotonicity, and equal share of extra costs. Cost monotonicity states that if a network connection cost increases, no agent should pay less. The idea behind equal share of extra costs is the following: consider a problem where the most expensive connection cost for any agent is the cost of connecting to the source. Additionally, the connection cost to the source is the same for all agents. If we assume that this connection cost increases by $x > 0$, then equal share of extra costs states that if agent i pays f_i in the original problem, he must pay $f_i + \frac{x}{n}$ when the cost increases (where n is the number of agents). In the second characterization of the folk rule they replace cost monotonicity by independence of irrelevant trees and population monotonicity by separability. Independence of irrelevant trees states that if two *mcstp* have the same minimum cost spanning tree with the same costs in each arc they belong to, then the rule must coincide in both *mcstp*. Assuming that two subsets of agents, S and $N \backslash S$, can be connected to the source either separately or jointly, separability states that if there are no savings when they are jointly connected to the source, then agents will pay the same in both circumstances.

In this paper we present two characterizations of the rule F in *mcstp* with groups, generalizing the results of Bergantiños and Vidal-Puga (2007a). We first adapt the properties to *mcstp* with groups. Cost monotonicity and independence of irrelevant trees do not need to be adapted (both properties make sense when there are groups). Equal share of extra costs is replaced by coalitional share of extra costs. Under the hypothesis of equal share of extra costs, coalitional share of extra costs states that the groups should share this extra cost x equally among them. Besides, the extra cost assigned to each group should be shared equally among the agents of this group. Population monotonicity is divided in two properties: population monotonicity over groups and population monotonicity over agents, as studied in Bergantiños and Gómez-Rúa (2010). Finally, separability is also divided in two properties: separability among groups and separability among agents.

The results establish that F is the unique rule in *mcstp* with groups satisfying cost monotonicity, population monotonicity over agents, population monotonicity over groups, and coalitional share of extra costs. We also prove that F is the unique rule in *mcstp* with groups satisfying independence of irrelevant trees, separability among agents, separability among groups, and coalitional share of extra costs; and finally we obtain that the properties used in both results are independent.

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