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SHAPLEY RULE, CORE AND MONOTONICITY

Francesc Carreras[1] and Guillermo Owen[2]

[1]Dept. of Applied Mathematics II and Industrial and Aeronautical Engineering School of Terrassa, Technical University of Catalonia. E–mail: francesc.carreras@upc.edu
[2]Department of Mathematics, Naval Postgraduate School, Monterey, California. E–mail: gowen@nps.edu

ABSTRACT

Pure bargaining problems with transferable utility are considered. By associating a quasi–additive cooperative game with each one of them, a *Shapley rule* for this class of problems is derived from the Shapley value for games. The analysis of this new rule includes its relationship with the core notion and standard monotonicity conditions.

Keywords: pure bargaining problem, Shapley value, core, monotonicity.

1. INTRODUCTION

The proportional rule has a long tradition in collective problems where some kind of monetary utility (costs, profits, savings) is to be shared among the agents. However, while its (apparent) simplicity might seem a reason for applying it in pure bargaining affairs, where only the whole and the individual utilities matter, its behavior is, in fact, questionable. A consistent alternative, the *Shapley rule*, is suggested as a much better solution for this kind of problems. Utilities will be assumed to be completely transferable, so that the class of problems considered here differs from the class of “bargaining problems” more commonly analyzed in the literature.

2. PURE BARGAINING PROBLEMS AND SHARING RULES

Let $N = \{1, 2, \ldots, n\}$ (with $n \geq 2$) be a set of agents and assume that there are given: (a) a set of utilities $u_1, u_2, \ldots, u_n$ available to the agents individually and (b) a total utility u_N that, alternatively, the agents can jointly get if all of them agree (utilities denoting costs will be represented by negative numbers). A vector $u = (u_1, u_2, \ldots, u_n|u_N)$ collects all this information and we will say that it represents a *pure bargaining problem* (*PBP*, in the sequel) on N. The *surplus* of u is defined as

$$\Delta(u) = u_N - \sum_{j \in N} u_j.$$

The problem consists in dividing u_N among the agents in a rational way, i.e. in such a manner that all of them should agree and feel (more or less) satisfied with the outcome. Of course, the individual utilities $u_1, u_2, \ldots, u_n$ should be taken into account.

Let E_{n+1} denote the $(n+1)$–dimensional vector space formed by all PBPs on N. In order to deal with, and solve, all possible PBPs on N, one should look for a *sharing rule*, i.e. a function f assigning to each PBP u a n–dimensional vector $f[u] = (f_1[u], f_2[u], \ldots, f_n[u])$. For each $i \in N$ the i–coordinate $f_i[u]$ will provide the share of u_N that corresponds to agent i according to f.

3. CLOSURES AND QUASI–ADDITIVE GAMES

The Shapley value [6], denoted here by φ, cannot be directly applied to PBPs as a sharing rule. We will therefore associate a TU (i.e., side–payment) cooperative game with each PBP in a natural way. Let $\mathcal{G}_N$ be the vector space of all cooperative TU games with N as set of players and let us define a map $\sigma : E_{n+1} \longrightarrow \mathcal{G}_N$ as follows. If $u = (u_1, u_2, \ldots, u_n | u_N)$ then $\overline{u} = \sigma(u)$ is given by

$$\overline{u}(S) = \sum_{i \in S} u_i \quad \text{if} \quad S \neq N \qquad \text{and} \qquad \overline{u}(N) = u_N.$$

The idea behind this definition is simple. Since, given a PBP u, nothing is known about the utility available to each intermediate coalition $S \subset N$ with $|S| > 1$, a reasonable assumption is that such a coalition can get the sum of the individual utilities of its members. Game $\overline{u}$ will be called the *closure* of u. It is not difficult to see that σ is a linear map and $\ker\{\sigma\} = \{0\}$, so that σ is one–to–one.

Let us recall that a cooperative game v is *additive* iff $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$. If we drop this condition just for $S = N$ and give the name *quasi–additive* to the games that fulfill it for all $S \subset N$, it follows that these games precisely form the image of σ, and hence a game is quasi–additive iff it is the closure of a PBP, which is unique. The dimension of the subspace of quasi–additive games is $n+1$. (If $n = 2$ then σ is onto and therefore *any* cooperative 2–person game is the closure of a PBP.)

As σ is an embedding of E_{n+1} into $\mathcal{G}_N$, reasonable restrictions for games can be adapted to PBPs after identifying each PBP with the corresponding closure. Then, we call to a PBP $u \in E_{n+1}$

- *additive (or inessential)* iff $u_1 + u_2 + \ldots + u_n = u_N$, i.e. iff $\Delta(u) = 0$;
- *superadditive* iff $u_1 + u_2 + \ldots + u_n < u_N$, i.e. iff $\Delta(u) > 0$;
- *symmetric* iff $u_i = u_j$ for all $i, j \in N$.

4. CORE AND THE SHAPLEY RULE

By applying the Shapley value to any quasi–additive game, i.e. to $\overline{u}$ for any $u \in E_{n+1}$, for each $i \in N$ we get

$$\varphi_i[\overline{u}] = u_i + \frac{\Delta(u)}{n}.$$

A quasi–additive game $\overline{u} = \sigma(u)$ is convex [7], that is, satisfies

$$\overline{u}(S) + \overline{u}(T) \leq \overline{u}(S \cap T) + \overline{u}(S \cup T) \quad \text{for all } S, T \subseteq N,$$

iff $\Delta(u) \geq 0$. The core [3] takes here the much simpler form

$$C(\overline{u}) = \{x = (x_1, x_2, \ldots, x_n) : \sum_{i \in N} x_i = u_N \text{ and } x_i \geq u_i \text{ for all } i = 1, 2, \ldots, n\}.$$

$C(\overline{u})$ is nonempty iff $\Delta(u) \geq 0$, and then the Shapley value $\varphi[\overline{u}] \in C(\overline{u})$ according to [7]. In fact, for a quasi–additive game the nonnegativity of the surplus is equivalent to, and not only sufficient for, the nonemptiness of the core.

The *disagreement point* is given by $x_1 = u_1, \ x_2 = u_2, \ \ldots, \ x_n = u_n$. The core is the simplex defined by $x_1 \geq u_1, \ x_2 \geq u_2, \ \ldots, \ x_n \geq u_n$ in the hyperplane $x_1 + x_2 + \ldots + x_n = u_N$. It becomes empty if $\Delta(u) < 0$ and reduces to the disagreement point if $\Delta(u) = 0$. Otherwise, that is, whenever $\Delta(u) > 0$, *the Shapley value $\varphi[\overline{u}]$ is the orthogonal projection of the disagreement point onto the core*, i.e. the intersection of the core with the orthogonal line $x_1 - u_1 = x_2 - u_2 = \ldots = x_n - u_n$.

We find here, thus, a particular case of Nash's classical bargaining problem [5]. The feasible set S is defined by $\sum_{i \in N} x_i \leq u_N$, the Pareto frontier is given by $\sum_{i \in N} x_i = u_N$, and the disagreement point is $D = (u_1, u_2, \ldots, u_n)$, which may lie above the Pareto frontier (just in case that $\Delta(u) < 0$). Moreover, $\varphi[\overline{u}]$ coincides with the Nash solution.

We are now ready to introduce the Shapley rule for PBPs. By setting

$$\overline{\varphi}[u] = \varphi[\overline{u}] \quad \text{for all } u \in E_{n+1}$$

we obtain a function $\overline{\varphi}$ that will be called the *Shapley rule* (for PBPs). It is given by

$$\overline{\varphi}_i[u] = u_i + \frac{\Delta(u)}{n} \quad \text{for each } i \in N \text{ and each } u \in E_{n+1}, \tag{1}$$

and hence it solves each PBP in the following way: (a) first, each agent is allocated his individual utility; (b) once this has been done, the remaining utility is equally shared among all agents. Notice that $\overline{\varphi}$ is linear.

The Shapley rule shows an "egalitarian flavor". Indeed, from Eq. (1) it follows that the Shapley rule is a mixture consisting of a "competitive" component, which rewards each agent according to the individual utility, and a "solidarity" component that treats all agents equally when sharing the surplus. Thus it satisfies standardness for two–agent PBPs in the sense of [4].

Remark 4.1 Our closure definition might seem rather pessimistic. For each $S \subset N$, the aspiration of the agents of S as a team in $\overline{u}$ is reduced to obtaining the sum of their individual utilities.

An alternative closure notion could be adopted by following the approach suggested in e.g. [1]. Let us define a second closure $\overline{u}^*$ of any PBP u by

$$\overline{u}(\emptyset) = 0 \qquad \text{and} \qquad \overline{u}^*(S) = u_N - \sum_{j \notin S} u_j \quad \text{if} \quad S \neq \emptyset.$$

This second closure notion assumes that the agents j not in S will be satisfied by receiving just u_j each, and the remaining utility will therefore be allocated to the members of S.

Let us see which would be the Shapley rule associated with this second closure notion. First, for any PBP u, $\delta = \overline{u}^* - \overline{u}$ is clearly given by $\delta(S) = \Delta(u)$ if $S \neq N$ and $\delta(N) = 0$. As δ is a symmetric game and $\delta(N) = 0$, it follows that $\varphi[\overline{u}^* - \overline{u}] = \varphi[\delta] = 0$, and hence $\varphi[\overline{u}^*] = \varphi[\overline{u}]$ for all $u \in E_{n+1}$. The conclusion is that the two Shapley rules (Eq. (1) and this remark, resp.) coincide.

To close this section we determine the set of PBPs where the Shapley rule and the proportional rule coincide and discuss individual rationality.

Proposition 4.2 *The Shapley rule and the proportional rule coincide on a PBP $u \in E^{\pi}_{n+1}$ iff u is additive or symmetric.*

Remark 4.3 For any sharing rule f, the property of *individual rationality* states that

$$f_i[u] \geq u_i \quad \text{for all } i \in N \text{ and all } u \in E_{n+1}.$$

It is easy to verify that the Shapley rule satisfies this property just in the domain of all additive or superadditive PBPs. Indeed, using Eq. (1) it follows that, for all $i \in N$,

$$\overline{\varphi}_i[u] \geq u_i \quad \text{iff} \quad \Delta(u) \geq 0.$$

5. AXIOMATIC CHARACTERIZATIONS OF THE SHAPLEY RULE

When looking for a function from E_{n+1} to the Euclidean n–space, some reasonable properties should be imposed. To state a set of them, we previously define dummy agent, null agent, and symmetric agents in a PBP. Agent $i \in N$ is a *dummy* in a PBP u iff $u_N = u_i + \sum_{j \neq i} u_j$, and *null* if, moreover, $u_i = 0$. Agents $i, j \in N$ are *symmetric* in a PBP u iff $u_i = u_j$. A PBP with a dummy agent must be additive and therefore all agents are dummies. Conversely, if a PBP is additive (i.e., inessential) then all agents are dummies.

Let us consider the following properties, stated (for a function f defined) on E_{n+1}:

(i) *Efficiency*: $\sum_{i \in N} f_i[u] = u_N$ for every u.

(ii) *Dummy agent property*: if i is a dummy in u then $f_i[u] = u_i$.

(iii) *Symmetry*: if i and j are symmetric in u then $f_i[u] = f_j[u]$.

(iv) *Additivity*: $f[u + v] = f[u] + f[v]$ for all u, v.

Efficiency means that the agents are going to share the total amount available to them. The dummy agent property essentially says that if a PBP is additive then each agent should receive his individual utility. Symmetry establishes that two agents that are equally powerful individually should receive the same payoff. Finally, additivity implies that the allocation in a sum of PBPs must coincide with the sum of allocations in each PBP.

Theorem 5.1 *(First main axiomatic characterization of the Shapley rule) There is one and only one function from E_{n+1} to the Euclidean n–space that satisfies properties (i)–(iv). It is the Shapley rule $\overline{\varphi}$.*

Several subsets of E_{n+1} deserve special attention, and it would be therefore of interest to have axiomatic characterizations for (the restriction of) the Shapley rule on each one of these domains. So as not to enlarge the analysis too much, we will consider only the following ones: (a) the domain of the proportional rule; (b) the open orthant of positive PBPs; (c) the open orthant of negative PBPs; (d) the open cone of superadditive PBPs; (e) the intersection of the cone of superadditive PBPs with the closed orthant of nonnegative PBPs; and (f) the intersection of the cone with the closed orthant of nonpositive PBPs.

If E denotes any of these subdomains of E_{n+1}, properties (i)–(iv) make sense for any function f defined on E if we state them only for $u, v \in E$. The sole exception is property (iv) for (a) since it is the only one of these domains not closed under addition of PBPs. Therefore in this case we will assume that additivity means: if $u, v, u+v \in E$ then $f[u+v] = f[u] + f[v]$.

Theorem 5.2 *(Additional axiomatic characterizations of the Shapley rule) If E denotes any of the subdomains of E_{n+1} mentioned just above, there is one and only one function f on E that satisfies properties (i)–(iv). In all cases it is (the restriction of) the Shapley rule $\overline{\varphi}$.*

In the literature on cooperative games, different monotonicity conditions have been suggested for solution concepts. Here we will recall some of the most relevant ones and will adapt them to the PBP setup, i.e., for sharing rules.

They are based on: (a) the classical *desirability relation* D for the players of any game $\overline{u}$ given by

$$iDj \text{ in } \overline{u} \qquad \text{iff} \qquad \overline{u}(S \cup \{i\}) - \overline{u}(S) \geq \overline{u}(S \cup \{j\}) - \overline{u}(S) \text{ for all } S \subseteq N \backslash \{i, j\},$$

which compares *the positions of two players in a common game*; and (b) a similar relation B, introduced in [2] and given by

$$\overline{u} \, B \, \overline{v} \text{ for } i \qquad \text{iff} \qquad \overline{u}(S \cup \{i\}) - \overline{u}(S) \geq \overline{v}(S \cup \{i\}) - \overline{v}(S) \text{ for all } S \subseteq N \backslash \{i\},$$

which compares *the positions of a common player in two games.* These conditions are:

- *monotonicity*: if iDj in $\overline{u}$ then $g_i[\overline{u}] \geq g_j[\overline{u}]$
- *strict monotonicity*: if iDj and $j \not{D} \, i$ in $\overline{u}$ then $g_i[\overline{u}] > g_j[\overline{u}]$
- *strong monotonicity [8]*: if $\overline{u} \, B \, \overline{v}$ for i then $g_i[\overline{u}] \geq g_i[\overline{v}]$
- *strict strong monotonicity*: if $\overline{u} \, B \, \overline{v}$ and $\overline{v} \not{B} \, \overline{u}$ for i then $g_i[\overline{u}] > g_i[\overline{v}]$.

The Shapley value satisfies all these conditions [2]. Thus, relations D and B, as well as the four monotonicity conditions stated above, make sense in PBPs (just replacing each quasi–additive game $\overline{u}$ with the corresponding PBP u given by σ^{-1}), and the Shapley rule $\overline{\varphi}$ satisfies all these conditions.

Once within the PBP framework, we obtain a new main axiomatic characterization of the Shapley rule on E_{n+1} that is quite different from Theorem 5.1.

Theorem 5.3 *(Second main axiomatic characterization of the Shapley rule) There is one and only one function f on E_{n+1} that satisfies efficiency, symmetry and strong monotonicity. It is the Shapley rule $\overline{\varphi}$.*

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