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MULTILINEAR EXTENSIONS AND SEMIVALUES RELATED THROUGH POTENTIAL MAPS

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RESUMO

Given a cooperative game defined in a set with n players and a semivalue defined on games with n players by means of its weighting vector, this work shows as the multilinear extension of the cooperative game can be modified according to the weights of the considered semivalue, so that the allocation to each player is obtained following an easy procedure based on classic steps of computation in the context of cooperative games.

Keywords: Cooperative game; Semivalue; Potential; Multilinear extension.

1. INTRODUCTION

The notion of potential is defined for any semivalue acting on the vector space of all cooperative games on a given set of players N. The potential induces an endomorphism of the space of cooperative games: the image games assign to every coalition the potential of the restricted games to the own coalition. The present work states the role of eigenvector of the unanimity games with specific eigenvalues depending on the considered semivalue. As a consequence, the computation of allocations to the players can be obtained from the multilinear extension by using a common procedure for all semivalues. This method generalizes the results obtained for the Shapley and Banzhaf values by Owen ([3], [4]).

2. COOPERATIVE GAMES AND SEMIVALUES

A *cooperative game* with transferable utility or TU game is a pair (N, v), where N is a finite set of *players* and $v : 2^N \to \mathbb{R}$ is the so-called *characteristic function*, which assigns to every *coalition* $S \subseteq N$ a real number $v(S)$, the *worth* of coalition S, and satisfies the natural condition $v(\emptyset) = 0$.

By G_N we denote the set of all TU games on N. For a given set of players N, we identify each game (N, v) with its characteristic function v. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subset T$, and *simple* if, moreover, $v(S) = 0$ or 1 for every $S \subseteq N$. A game v is *additive* if $v(S \cup T) = v(S) + v(T)$ whenever $S \cap T = \emptyset$. Given a nonempty coalition $T \subseteq N$, the *restriction* to T of a given game v on N is the game $v_{|T}$ on T that we will call a *subgame* of v and is defined by

$$v_{|T}(S) = v(S) \quad \text{for all } S \subseteq T.$$

Endowed with the natural operations, $v + v'$ and λv for $\lambda \in \mathbb{R}$, the set of all cooperative games G_N is a *vector space*. For every nonempty coalition $T \subseteq N$ the *unanimity game* u_T is defined on N by

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

The set of all unanimity games is a basis for G_N, so that $dim(G_N) = 2^n - 1$ if $n = |N|$.

The *multilinear extension* [3] (MLE, in the sequel) of a game $v \in G_N$ is the function $f_v : [0,1]^n \longrightarrow \mathbb{R}$ defined by

$$f_v(X_N) = \sum_{S \subseteq N} \prod_{i \in S} x_i \prod_{j \in N \setminus S} (1 - x_j) v(S),$$

where X_N denotes the set of variables x_i for $i \in N$.

If $\emptyset \neq T \subseteq N$, then $\quad f_{u_T}(X_N) = \prod_{i \in T} x_i.$

If $v = \sum_{T \subseteq N:\, T \neq \emptyset} \alpha_T u_T$, then $\quad f_v(X_N) = \sum_{T \subseteq N:\, T \neq \emptyset} \alpha_T \prod_{i \in T} x_i. \qquad (\S)$

By a *value* on G_N we will mean a map $f : G_N \to \mathbb{R}^N$, which assigns to every game v on N a vector $f[v]$ with components $f_i[v]$ for all $i \in N$.

Following Dubey, Neyman and Weber's [2] axiomatic description, a value $\psi : G_N \to \mathbb{R}^N$ is a *semivalue* iff it satisfies the following properties:

(i) *Linearity*: $\psi[\lambda v + \mu w] = \lambda \psi[v] + \mu \psi[w]$ for all $v, w \in G_N$ and $\lambda, \mu \in \mathbb{R}$.
(ii) *Anonymity*: $\psi_{\theta i}[\theta v] = \psi_i[v]$ for all θ permutation on N, $i \in N$, and $v \in G_N$.
(iii) *Positivity*: if v is monotonic, then $\psi[v] \geq 0$.
(iv) *Projection property*: if v is additive, then $\psi_i[v] = v(\{i\})$ for all $i \in N$.

The same authors provided a useful characterization of semivalues by means of *weighting coefficients*: (a) for every nonnegative *weighting vector* $(p_s)_{s=1}^n$ such that

$$\sum_{s=1}^{n} \binom{n-1}{s-1} p_s = 1,$$

the expression

$$\psi_i[v] = \sum_{S \subseteq N:\, i \in S} p_s [v(S) - v(S \setminus \{i\})] \quad \text{for all } i \in N \text{ and all } v \in G_N,$$

where $s = |S|$, defines a semivalue ψ; (b) conversely, every semivalue can be obtained in this way; (c) the correspondence $(p_s)_{s=1}^n \mapsto \psi$ is bijective.

Well known examples of semivalues are the *Shapley value* [5], for which $p_s = 1/n\binom{n-1}{s-1}$, and the *Banzhaf value* [4], for which $p_s = 1/2^{n-1}$.

A semivalue $\psi = \psi^n$ defined on cooperative games with n players gives rise to induced semivalues ψ^t on cooperative games with t players, where $t = n-1, \ldots, 1$, recurrently defined by their weighting coefficients obtained according to the *Pascal triangle (inverse) formula:*

$$p_s^t = p_s^{t+1} + p_{s+1}^{t+1} \quad \text{for} \quad 1 \leq s \leq t < n.$$

3. POTENTIAL MAPS AND MULTILINEAR EXTENSIONS

Definition 1. [1] Let ψ be a semivalue on G_N defined by the weighting vector $(p_s^n)_{s=1}^n$. The ψ–potential for the restriction of a given game $v \in G_N$ to a nonempty coalition $T \subseteq N$ is defined by

$$P_\psi(v_{|T}) = \sum_{S \subseteq T: S \neq \emptyset} p_s^t v(S)$$

Definition 2. [1] Let ψ be a semivalue on G_N defined by the weighting vector $(p_s^n)_{s=1}^n$. The ψ–potential game of v is the game $P_\psi^*(v)$ defined by

$$P_\psi^*(v)(\emptyset) = 0 \quad \text{and} \quad P_\psi^*(v)(S) = P_\psi(v_{|S}) \quad \text{if} \quad \emptyset \neq S \subseteq N.$$

The endomorphism P_ψ^* of G_N that transforms any game v in the game $P_\psi^*(v)$ will be called the ψ–potential map.

Theorem 1. Let ψ be a semivalue on G_N defined by the weighting vector $(p_s^n)_{s=1}^n$. Every unanimity game u_T is an eigenvector for the mapping P_ψ^* with eigenvalue p_t^t:

$$P_\psi^*(u_T) = p_t^t \, u_T \quad \text{if} \quad \emptyset \neq T \subseteq N.$$

Theorem 2. Let ψ be a semivalue on G_N defined by the weighting vector $(p_s^n)_{s=1}^n$ and let the MLE of $v \in G_N$ be given by Eq. (§). Then:

(i) The MLE of game $P_\psi^*(v)$ is: $\quad f_{P_\psi^*(v)}(X_N) = \sum_{T \subseteq N:\, T \neq \emptyset} p_t^t \alpha_T \prod_{i \in T} x_i.$

(ii) If $\emptyset \neq T \subseteq N$, the allocation to player $i \in T$ in the subgame $v_{|T}$ given by the induced semivalue ψ^t is

$$\psi_i^t[v_{|T}] = \left. \frac{\partial f_{P_\psi^*(v)}(X_T, 0_{N \setminus T})}{\partial x_i} \right|_{1_T}.$$

Example 1. Let us consider the *weighted majority game* $[68; 46, 42, 23, 15, 9]$, i.e., the 5–person (simple) game v defined by $v(S) = 1$ if $\sum_{i \in S} w_i \geq q$ (winning coalitions) and $v(S) = 0$ otherwise, where $q = 68$ is the *quota* and the *weights* are $w_1 = 46$, $w_2 = 42$, and so on up to $w_5 = 9$ (incidentally, this game describes the Catalonian Parliament during Legislature 2003–2007). The set of minimal winning coalitions in this game is $W^m(v) = \{\{1,2\},\{1,3\},\{1,4,5\},\{2,3,4\},\{2,3,5\}\}$ and its MLE is

$$f_v(x_1,\dots,x_5) = x_1x_2 + x_1x_3 - x_1x_2x_3 + x_1x_4x_5 + x_2x_3x_4 + \\ + x_2x_3x_5 - \sum_{i<j<k<\ell} x_ix_jx_kx_\ell + 2x_1x_2x_3x_4x_5.$$

Let us consider the semivalue $\psi = \psi^5$ defined by the weighting vector

$$(p_s^5)_{s=1}^5 = (5/48, 1/12, 1/16, 1/24, 1/48).$$

We first compute the last weighting coefficients for its induced semivalues:

$$p_5^5 = 1/48, \quad p_4^4 = 1/16, \quad p_3^3 = 1/6, \quad p_2^2 = 5/12 \quad \text{and} \quad p_1^1 = 1.$$

According to Theorem 2 (i), these amounts allow us to obtain the multilinear extension of game $P^*_\psi(v)$, habitually called *modified MLE according to semivalue* ψ:

$$f_{P^*_\psi(v)}(x_1,\dots,x_5) = \frac{5}{12}\,[x_1x_2 + x_1x_3] + \frac{1}{6}\,[-x_1x_2x_3 + x_1x_4x_5 + x_2x_3x_4 + \\ + x_2x_3x_5] - \frac{1}{16}\sum_{i<j<k<\ell} x_ix_jx_kx_\ell + \frac{1}{24}\,x_1x_2x_3x_4x_5.$$

Now, a second step (part (ii) in Theorem 2) allows us compute allocations to the players in game v. For it, after partial differentiation, we replace the remaining variables with 1. For instance, for player 1:

$$\frac{\partial f_{P^*_\psi(v)}}{\partial x_1}(x_1,\dots,x_5) = \frac{5}{12}\,[x_2+x_3] + \frac{1}{6}\,[-x_2x_3+x_4x_5] - \frac{1}{16}\sum_{1<i<j<k} x_ix_jx_k + \frac{1}{24}\,x_2x_3x_4x_5,$$

$$\psi_1^5[v] = \frac{\partial f_{P^*_\psi(v)}}{\partial x_1}(1_N) = \frac{5}{8}.$$

Following a similar procedure, the allocations to the remaining players in game v can be computed. Thus, the payoff vector to players in game v according to semivalue $\psi = \psi^5$ is

$$\psi^5[v] = (\ 5/8,\ 3/8,\ 3/8,\ 1/8,\ 1/8\ ).$$

4. CONCLUSION

The main result of the paper is a theoretical property proved for the unanimity games in Theorem 1. Since all cooperative games are linear combination of unanimity games, the result can be adapted to the MLE of the games, so that, the MLE of the image game by the potential map is obtained by multiplying each product of t variables in the MLE of the original game by the corresponding coefficient. This is the first step in the procedure to compute allocations by any semivalue (part (i) in Theorem 2). The second step (part (ii)) is standard: to obtain the allocation to a player i according to the selected semivalue, compute the partial derivative of the modified MLE with respect to the variable x_i and, after, replace the variables with 1.

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