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COMPUTING MAXIMAL DUAL-FEASIBLE FUNCTIONS USING A NOVEL APPROACH

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ABSTRACT

Dual-feasible functions have been used in the broad area of combinatorial optimization to compute bounds and cutting planes for integer programming problems. These functions have been rediscovered recently, and studied in depth in different articles published in the literature. In this paper, we propose a new strategy for building families of staircase maximal dual-feasible functions using the properties of the dual solutions of a related linear programming problem. We describe the foundations of the strategy and we provide an example of a dual-feasible function generated in this way. Our study is completed by a comparison of this function with the best functions proposed in the literature.

Keywords: Integer programming, dual-feasible functions, maximality.

1. INTRODUCTION

A *dual-feasible function (DFF)* $f : [0,1] \rightarrow [0,1]$ is a function for which for any finite set $\{x_i \in \mathbb{R}_+ : i \in I\}$ of nonnegative values, the following holds

$$\sum_{i \in I} x_i \leq 1 \Longrightarrow \sum_{i \in I} f(x_i) \leq 1. \tag{1}$$

These functions have been used in the past to compute bounds and valid inequalities for integer programming problems with knapsack constraints. A survey describing the best functions of the literature was proposed recently in [Clautiaux *et al.*, 2010].

A DFF f is said to be *maximal* (MDFF), if there is no other DFF whose values are always greater than or equal to those of f for the same values of the domain, *ie* there is no DFF $g : [0,1] \rightarrow [0,1]$ such that $g(x) \geq f(x)$, for all $x \in [0,1]$. A DFF f is a MDFF if it is symmetric and superadditive. In [Rietz *et al.*, 2010], the authors proposed a simplified way of proving the maximality of a dual-feasible function which is reproduced in following theorem.

Theorem 1 : A function $f : [0,1] \rightarrow \mathbb{R}_+$ is a MDFF if:

$$\begin{aligned} f(0) &= 0 && (2)\\ 1 &= f(x) + f(1-x)\ \forall x \in [0, 1/2] && (3)\\ f(x_1 + x_2) &\geq f(x_1) + f(x_2)\ \forall x_1, x_2 \text{ with } 0 < x_1 \leq x_2 < \frac{1}{2} \wedge x_1 + x_2 \leq \frac{2}{3} && (4) \end{aligned}$$

The condition (4) is equivalent to the following: for any $x_1, x_2, x_3 \in \mathbb{R}$ with

$$0 < x_1 \leq x_2 \leq x_3 \wedge x_1 + x_2 + x_3 = 1 \qquad (5)$$

the inequality $f(x_1) + f(x_2) + f(x_3) \leq 1$ holds. Condition (5) implies $x_3 \geq 1/3$, $x_2 < 1/2$ and $x_1 \leq 1/3$.

In this paper, we describe a new strategy for deriving maximal dual-feasible functions using the properties of the dual solutions of a related linear programming problem. This problem corresponds in fact to the linear relaxation of the column generation formulation of the 1-dimensional cutting stock problem proposed first in [Gilmore and Gomory, 1961]. Let $E := (m; L; \mathbf{l}; \mathbf{b})$ be an instance of the 1-dimensional cutting stock problem, with m representing the number of items, L the size of the rolls, $\mathbf{l}$ the vector of item sizes, and $\mathbf{b}$ the item demands. This formulation stands as follows.

$$\begin{aligned} \min \quad & \sum_{p \in P} \lambda_p && (6)\\ \text{s.t.} \quad & \sum_{p \in P} a_{ip}\lambda_p \geq b_i,\ i = 1, \ldots, m, && (7)\\ & \lambda_p \geq 0, \text{ and integer, } p \in P. && (8) \end{aligned}$$

A column in this model is associated to a feasible cutting pattern in the set P. The variables λ_p determines the number of times a pattern p is used in the solution, while the coefficients a_{ip} denote the number of items i cut from the pattern p. After solving the continuous relaxation of (6)-(8) for E with the revised simplex method to optimality, the vector of dual multipliers is denoted by $\mathbf{d}$. Then, for any feasible pattern with a vector $\mathbf{a}$ of coefficients a_{ip}, $i = 1, \ldots, m$, it holds that $\mathbf{d}^T\mathbf{a} \leq 1$. If the corresponding pattern p is used in a positive quantity in the basic solution, then $\mathbf{d}^T\mathbf{a} = 1$. This fact can be used to derive new MDFF as we will show in the following section. We will show also that the functions derived in this way are different from others already proposed in the literature. For the sake of completeness, we give below the definition of the best functions to which our general scheme for generating DFF will be compared.

- $f_{FS,1}$ from [Fekete and Schepers, 2001], with $k \in \mathbb{N} \setminus \{0\}$:

$$f_{FS,1}(x;k) = \begin{cases} x, & \text{if } (k+1) * x \in \mathbb{N}, \\ \lfloor (k+1)x \rfloor / k, & \text{otherwise.} \end{cases}$$

- $f_{CCM,1}$ from [Carlier *et al.*, 2007], with $C \in \mathbb{R}$ and $C \geq 1$:

$$f_{CCM,1}(x; C) = \begin{cases} \lfloor Cx \rfloor / \lfloor C \rfloor, & \text{if } x < 1/2, \\ 1/2, & \text{if } x = 1/2, \\ 1 - f_{CCM,1}(1-x), & \text{if } x > 1/2. \end{cases}$$

- $f_{VB,2}$ adapted from a function of [Vanderbeck, 2000], with $k \in \mathbb{N} \setminus \{0, 1\}$:

$$f_{VB,2}(x; k) = \begin{cases} \max\{0, \lceil kx \rceil - 1\} / (k-1), & \text{if } x < 1/2, \\ 1/2, & \text{if } x = 1/2, \\ 1 - f_{VB,2}(1-x), & \text{for } x > 1/2. \end{cases}$$

- $f_{LL,2}$ from [Letchford and Lodi, 2002], with $C \in \mathbb{R} \setminus \mathbb{N}$, $C > 1$ and $k \in \mathbb{N}, k \geq \lceil 1/\mathsf{frac}(C) \rceil - 1$:

$$f_{LL,2}(x; C, k) = \begin{cases} \left(\lfloor Cx \rfloor + \max\left\{0, \left\lceil (k-1) * \frac{\mathsf{frac}(Cx) - \mathsf{frac}(C)}{1 - \mathsf{frac}(C)} \right\rceil / k \right\} \right) / \lfloor C \rfloor, & \text{if } x < \frac{1}{2}, \\ 1/2, & \text{if } x = \frac{1}{2}, \\ 1 - f_{LL,2}(1-x), & \text{if } x > \frac{1}{2} \end{cases}$$

- $f_{DG,1}(x; C, k)$ from [Dash and Günlük, 2006], with $C \in \mathbb{R} \setminus \mathbb{N}, C > 1$ and $k \in \mathbb{N}$, $k \geq \left\lceil \frac{1}{\mathsf{frac}(C)} \right\rceil - 1$:

$$f_{DG,1} = \frac{\lfloor Cx \rfloor}{\lfloor C \rfloor} + \frac{1}{\lfloor C \rfloor} * \begin{cases} \frac{\mathsf{frac}(Cx) - \mathsf{frac}(C)}{1 - \mathsf{frac}(C)}, & \text{if } (k-1) * \frac{\mathsf{frac}(Cx) - \mathsf{frac}(C)}{1 - \mathsf{frac}(C)} \in \mathbb{N}, \\ \max\left\{0, \left\lceil (k-1) * \frac{\mathsf{frac}(Cx) - \mathsf{frac}(C)}{1 - \mathsf{frac}(C)} \right\rceil \right\} / k, & \text{otherwise.} \end{cases}$$

2. NEW MAXIMAL DUAL-FEASIBLE FUNCTIONS

Consider the linear relaxation of the column generation model (6)-(8) introduced in the previous section and the corresponding instance $E := (m; L; \mathbf{l}; \mathbf{b})$ of the 1-dimensional cutting stock problem. The fact that $\mathbf{d}^T \mathbf{a} = 1$ in a basic solution for the cutting pattern related to $\mathbf{a}$ yields an interesting starting point for deriving a DFF f that is maximal. Let $f(\ell_i/L) = d_i$. This implies $\sum_{x \in I} f(x) \leq 1$, if I is any finite set of numbers ℓ_i/L with a sum not above 1. To build a MDFF from this approach, it is necessary to set $f(0.5) := 0.5$, and to define the other function values such that the conditions for maximality are fulfilled. The following example shows how this can be done for a small instance E.

Example: Let $E := (3; 9; (5, 3, 2)^T; (1, 2, 1)^T)$ be an instance of the 1-dimensional cutting stock problem. For this instance, we have $\mathbf{d} = (2/3, 1/3, 1/6)^T$. Therefore, the function f is defined as follows: $f(x) = 0$ for $0 \leq x < 2/9$, $f(x) = 1/6$ for $2/9 \leq x < 1/3$, $f(x) = 1/3$ for $1/3 \leq x \leq 4/9$, $f(x) = 1/2$ for $4/9 < x < 5/9$, $f(x) = 2/3$ for $5/9 \leq x \leq 2/3$, $f(x) = 5/6$ for $2/3 < x \leq 7/9$ and $f(x) = 1$ for $7/9 < x \leq 1$. This function f is a staircase function. Because shortening certain items in a given instance of the 1-dimensional cutting stock problem by a sufficient small value leads to an equivalent instance, it is possible to change some intervals for the obtained staircase function. Clearly, $f(x) = 1/6$ for an $x \leq 1/7$ is impossible. Nevertheless, setting $f(x) = 0$ for $0 \leq x < 1/6$, $f(x) = 1/6$ for $1/6 \leq x < 1/3$, $f(x) = 5/6$ for $2/3 < x \leq 5/6$, $f(x) = 1$ for $5/6 < x \leq 1$ and leaving f unchanged in the case $1/3 \leq x \leq 2/3$ yields another MDFF, because all conditions are fulfilled. This approach yields the family of functions that is presented next.

<u>*Assertion 1:*</u> Let $\alpha, \beta \in \mathbb{N}$ be two parameters with $\alpha \geq 1$ and $\beta \geq 2$. The following settings yield a maximal dual-feasible staircase function:

$$\begin{aligned}
\gamma &:= \lceil 1 + (\alpha+2) * (\beta-1)/2 \rceil = \beta + \lceil \alpha * (\beta-1)/2 \rceil && (9)\\
\delta &:= (\alpha+2) * (\gamma + 1 - \beta) - 1 \\
f(x) &:= \begin{cases} (\lfloor \frac{\delta * x}{\gamma} \rfloor + \frac{1}{\beta} * \max\{0, \lfloor \delta * x \rfloor \mod \gamma + \beta - \gamma\})/\alpha & \text{if } 0 \leq x < 1/2 \\ 1/2 & \text{if } \quad x = 1/2 \\ 1 - f(1-x) & \text{if } 1/2 < x \leq 1 \end{cases} && (10)
\end{aligned}$$

Proof: The function f is obviously a staircase function. The definition (10) implies $f(x) = 0$ for $\lfloor \delta * x \rfloor \leq \gamma - \beta$. Clearly, we have $\gamma > \beta$, since the definition (9) yields $\gamma - \beta \geq 1 + 3*(\beta-1)/2 - \beta = (\beta-1)/2 > 0$. To apply Lemma 3 of [Rietz *et al.*, 2010] for proving the maximality of a DFF, we resort to the following DFF $f_{BJ,1}$ described in [Burdett and Johnson, 1977]:

$$f_{BJ,1}(x;C) = \frac{1}{\lfloor C \rfloor} * \left(\lfloor Cx \rfloor + \max\left\{0, \frac{\mathsf{frac}(Cx) - \mathsf{frac}(C)}{1 - \mathsf{frac}(C)}\right\}\right),$$

with $C \in \mathbb{R}$ and $C \geq 1$. Set $C := \frac{\delta}{\gamma} = \alpha + 2 - \frac{(\alpha+2)*(\beta-1)+1}{\gamma} = \alpha + \frac{\lceil \alpha*(\beta-1)/2 \rceil * 2 - \alpha*(\beta-1)+1}{\gamma} \in (\alpha, \alpha+1)$, because the nominator of the last fraction is one or two and the denominator at least three, hence $\lfloor C \rfloor = \alpha$. Let $\tau := 1 - \beta/\gamma$. This yields

$$\frac{\mathsf{frac}(Cx) - \tau}{1 - \tau} = \frac{\mathsf{frac}(\delta * x/\gamma) - 1 + \beta/\gamma}{1 - 1 + \beta/\gamma} = \frac{\delta * x - \gamma * \lfloor \delta * x/\gamma \rfloor + \beta - \gamma}{\beta}.$$

Since $\lfloor \delta * x \rfloor \mod \gamma = \lfloor \delta * x \rfloor - \gamma * \lfloor \delta * x/\gamma \rfloor$, the estimation $f(x) \leq f_{BJ,1}(x)$ is true for all $x \in [0, 1/2)$. The function f is symmetric by construction (10). Because $f_{BJ,1}$ is a MDFF and $f_{BJ,1}(1) = f(1) = 1$, it follows from the lemma that f is a MDFF, if $f(x_1 + x_2) \geq f(x_1) + f(x_2)$ for all $x_1, x_2 \in (0, 1/2)$ with $x_1 + x_2 < 1/2$ is shown. For this purpose, let $h(x_1, x_2) := \alpha\beta * \Big(f(x_1 + x_2) - f(x_1) - f(x_2)\Big)$, $\delta * x_1 = a_1 * \gamma + b_1$ and $\delta * x_2 = a_2 * \gamma + b_2$ with $a_1, a_2 \in \mathbb{N}$ and $0 \leq b_1, b_2 < \gamma$. This yields

$$\begin{aligned}
h(x_1, x_2) &= \beta * (\lfloor \frac{\delta * (x_1 + x_2)}{\gamma} \rfloor - \lfloor \frac{\delta * x_1}{\gamma} \rfloor - \lfloor \frac{\delta * x_2}{\gamma} \rfloor) + \\
&\quad \max\{0, \lfloor \delta * (x_1 + x_2) \rfloor \mod \gamma + \beta - \gamma\} - \\
&\quad \max\{0, \lfloor \delta * x_1 \rfloor \mod \gamma + \beta - \gamma\} - \max\{0, \lfloor \delta * x_2 \rfloor \mod \gamma + \beta - \gamma\} \\
&= \beta * (a_1 + a_2 + \lfloor \frac{b_1 + b_2}{\gamma} \rfloor - a_1 - \lfloor \frac{b_1}{\gamma} \rfloor - a_2 - \lfloor \frac{b_2}{\gamma} \rfloor) +
\end{aligned}$$

$$
\begin{aligned}
& \max\{0, \lfloor b_1 + b_2 \rfloor \mod \gamma + \beta - \gamma\} - \\
& \max\{0, \lfloor b_1 \rfloor \mod \gamma + \beta - \gamma\} - \max\{0, \lfloor b_2 \rfloor \mod \gamma + \beta - \gamma\} \\
= \ & \beta * \lfloor \frac{b_1 + b_2}{\gamma} \rfloor + \max\{0, \lfloor b_1 + b_2 \rfloor \mod \gamma + \beta - \gamma\} - \\
& \max\{0, \lfloor b_1 \rfloor + \beta - \gamma\} - \max\{0, \lfloor b_2 \rfloor + \beta - \gamma\}.
\end{aligned}
$$

The three maxima are below β. If the second or third maximum vanishes, then $h(x_1, x_2) \geq 0$ is obvious. Otherwise it follows that $b_1, b_2 \geq 1 + \gamma - \beta$. Since $h(x_1, x_2)$ is integer, it is sufficient to show $h(x_1, x_2) > -1$. A small case distinction is made for this purpose:

1. If $b_1 + b_2 < \gamma$, then the first maximum is $\lfloor b_1 + b_2 \rfloor + \beta - \gamma$, since $\lfloor b_1 + b_2 \rfloor > b_1 + b_2 - 1 \geq 1 + 2\gamma - 2\beta$. Hence $h(x_1, x_2) > b_1 + b_2 - 1 + \beta - \gamma - (\lfloor b_1 \rfloor + \beta - \gamma) - (\lfloor b_2 \rfloor + \beta - \gamma) = b_1 + b_2 - 1 - \beta + \gamma - \lfloor b_1 \rfloor - \lfloor b_2 \rfloor > -1$ because $\beta < \gamma$.

2. $b_1 + b_2 \geq \gamma$ yields $h(x_1, x_2) = \beta + \max\{0, \lfloor b_1 + b_2 \rfloor + \beta - 2\gamma\} - (\lfloor b_1 \rfloor + \beta - \gamma) - (\lfloor b_2 \rfloor + \beta - \gamma) = \max\{0, \lfloor b_1 + b_2 \rfloor + \beta - 2\gamma\} + 2\gamma - \beta - \lfloor b_1 \rfloor - \lfloor b_2 \rfloor$. If the maximum is positive, then $h(x_1, x_2) = \lfloor b_1 + b_2 \rfloor - \lfloor b_1 \rfloor - \lfloor b_2 \rfloor \geq 0$. Otherwise, $\lfloor b_1 + b_2 \rfloor \leq 2\gamma - \beta$ implies $\lfloor b_1 \rfloor + \lfloor b_2 \rfloor \leq 2\gamma - \beta$ and $h(x_1, x_2) \geq 0$.

Because all prerequisites of Lemma 3 of [Rietz *et al.*, 2010] are fulfilled, the function f is a MDFF.

3. COMPARISON WITH THE FUNCTIONS OF THE LITERATURE

The non-equivalence to the functions $f_{FS,1}$, $f_{CCM,1}$, $f_{LL,2}$, $f_{BJ,1}$, $f_{DG,1}$ and $f_{VB,2}$ will be shown for the case $\alpha = \beta = 2$. This yields $\gamma = 3$, $\delta = 7$ and $f(x) = 0$ for all $x \in [0, 2/7)$, $f(x) = 1/4$ for all $x \in [2/7, 3/7)$ and $f(x) = 1/2$ for all $x \in [3/7, 4/7]$. Hence, f is continuous from the right in the interval $[0, 1/2)$. We show in the sequel that this staircase function is not equivalent to the ones referred above:

- $f_{FS,1}$: because this function is non-continuous for $(k+1)x \in \mathbb{N}$ only, but for $x \in (0,1) \wedge (k+1)x \in \mathbb{N}$ one has $\lceil (k+1)x \rceil / k = x * (k+1)/k > x$, hence $f_{FS,1}$ is at that places non-continuous from the right also and can therefore not be equivalent to f;

- $f_{VB,2}$ is also non-continuous from the right for $x = 1/k$;

- $f_{CCM,1}$ cannot yield the same as the function (10) for $\alpha = \beta = 2$. To prove this, we analyze first the point $2/7$. Since $f(x) = 0$ for $0 \leq x < 2/7$ but $f(2/7) = 1/4 > 0$, it follows that $\lambda = 7/2$. But then, we have $f_{CCM,1}(2/7) = 1/3 \neq 1/4$ as needed;

- $f_{LL,2}$ and $f_{DG,1}$ cannot be equivalent to the function (10) with $\alpha = \beta = 2$ as the discussion of the point $2/7$ shows. Because of $f(x) = 0$ for $0 \leq x < 2/7$, it follows that $\lambda \leq 7/2$. If $\lambda = 7/2$, then $f_{LL,2}(2/7) = f_{DG,1}(2/7) = 1/3 \neq f(2/7)$. If $\lambda < 7/2$, then $\lambda x < 1$ for all $x \leq 2/7$, such that $\mathsf{frac}(\lambda x)$ and therefore $f_{LL,2}(x)$ and $f_{DG,1}(x)$ are continuous from the left at $x = 2/7$. Hence, we have $f_{LL,2}(x) = f_{DG,1}(x) = 0$ for all $x \in [0, 2/7]$.

The proof of the fact that f is a MDFF shows also that γ and δ could be changed in a certain range. This yields functions that are somewhat different. The main influence comes from α and β. This is shown in the figure for the following cases: 1: $\alpha = 1$, $\beta = 2$, $\gamma = 3$, $\delta = 5$; 2: $\alpha = \beta = 2$, $\gamma = 3$, $\delta = 7$; 3: $\alpha = 2$, $\beta = 6$, $\gamma = 11$, $\delta = 23$; 4:$\alpha = 6$, $\beta = 2$, $\gamma = 5$, $\delta = 31$.

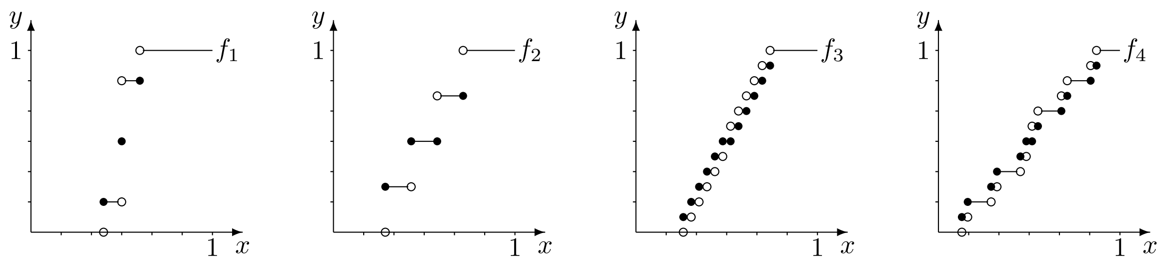

Figure 1: Examples for several parameter settings

4. CONCLUSIONS

We described an original approach for computing maximal staircase dual-feasible functions. A further step into the exploration of these functions will consist in finding strategies for computing good parameters with small complexity. This exercise raises many issues that will be explored in the future.

REFERENCES

Carlier, J., Clautiaux, F. and Moukrim A. (2007) New reduction procedures and lower bounds for the two-dimensional bin-packing problem with fixed orientation. Computers and Operations Research, 34:2223–2250.

Clautiaux, F., Alves, C. and Carvalho, J. (2010) A survey of dual-feasible functions for bin packing problems. Annals of Operations Research, 179:317-342.

Burdett,C. and Johnson, E. (1977) A subadditive approach to solve linear integer programs. Annals of Discrete Mathematics, 1:117-144.

Dash, S. and Günlük, O. Valid inequalities based on simple mixed-integer sets. Mathematical Programming, 105:29-53.

Fekete, S. and Schepers, J (2001) New classes of fast lower bounds for bin packing problems. Mathematical Programming, 91:11-31.

Gilmore, P. and Gomory, R. (1961) A linear programming approach to the cutting stock problem. Operations Research, 9:849-859.

Letchford, A. and Lodi, A. (2002) Strengthening Chvával-Gomory cuts and Gomory fractional cuts. Operations Research Letters, 30:74–82.

Rietz, J., Alves, C. and Carvalho, J. (2010) Theoretical investigations on maximal dual feasible functions. Operations Research Letters, 38:174–178.

Vanderbeck, F. (2000). Exact algorithm for minimizing the number of setups in the one-dimensional cutting stock problem. Operations Research, 46(6):915-926.